April 22, 2001.

Making Gregory Run.

Gregory's famous power series

$$\mathrm{Atn}(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \frac{z^9}{9} \cdots$$

can be an efficient and accurate means of calculating π when used in a formula of the type

$$\frac{\pi}{4} = n \cdot \mathrm{Atn}(r) - \mathrm{Atn}(s)\,, \tag{$*$}$$

where r and s are small (say, $\leq 1/2$) rational numbers with smallish denominators. If $\alpha = \mathrm{Atn}(r)$ and $\beta = \mathrm{Atn}(s)$, this says that $\tan(n\alpha - \beta) = 1$, i.e., $\tan(n\alpha) - \tan(\beta) = 1 + \tan(n\alpha)\tan(\beta)$, or

$$\tan(\beta) = \frac{\tan(n\alpha) - 1}{\tan(n\alpha) + 1}$$

where both $r = \tan\alpha$ and $s = \tan(\beta)$ are small with smallish denominators. To find candidates, we cast about for r and n such that $(1 + ri)^n = u + vi$ lies close to the diagonal $\Im(z) = \Re(z)$, in particular that $(v - u)/(v + u)$ be small and well-behaved. The easiest and best solutions to this problem seem to be

$$\frac{\pi}{4} = 2 \cdot \mathrm{Atn}(1/2) - \mathrm{Atn}(1/7) = 4 \cdot \mathrm{Atn}(1/5) - \mathrm{Atn}(1/239)$$

with the second a tad better than the first. Other expressions, such as

$$3 \cdot \mathrm{Atn}(1/4) + \mathrm{Atn}(5/99) \quad \text{or} \quad 5 \cdot \mathrm{Atn}(1/6) - \mathrm{Atn}(475/11767)$$

have much uglier values for s. The smallest value of s found in a somewhat desultory search with $n < 10$ occurred for $r = 7/71$ and $n = 8$. Though less than 0.0008 in size, it is arithmetically represented by $47,075,244,863$ over $59,328,860,591,809$ — in lowest terms.