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Sums of Consecutive Squares.

This is to prove the formula

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)}{2} + \frac{(n-1)n(n+1)}{3}\,. \tag{$*$}$$

We start by exploring the second term on the right. By an n-slab we shall mean a solid $n \times (n-1)$ rectangle of unit thickness; by an n-brick, a stack of $n+1$ such slabs. We shall cut three n-slabs off an n-brick to obtain an $(n-1)$-brick as follows.

1) Taking the top n-slab off the stack, we obtain a brick of depth $(n-1)$ with an $n \times n$ face (width and height).
2) Cutting an n-slab off at right angles to the width, we get a brick of depth $(n-1)$ with an $(n-1) \times n$ face.
3) Cutting an n-slab off at right angles to the depth, we get a brick of depth $(n-2)$ with an $(n-1) \times n$ face.

Now we can apply a similar process to the left-over $(n-1)$-brick: cut off three $(n-1)$-slabs to obtain an $(n-2)$-brick $\ldots$, and so on until we are down to a 2-brick (having dimensions $1 \times 2 \times 3$).

A moments reflection shows that we have proved

$$3 \cdot \sum_{k=1}^{n} (k-1)k = (n-1)n(n+1)\,, \tag{$\dagger$}$$

whence the desired formula via $1 + 2 + \cdots + n = (n+1)n/2$.

The whole process can be presented more visually as follows. First make a kind of pyramid (or ziggurat) of square slabs of unit thickness and areas $1, 4, 9, \ldots n^2$, all stacked on the same "origin". Then slice off a "face" having the form of a staircase of unit thickness cut from an $n \times n$ square (hence having volume $1 + 2 + \cdots + n$). Three copies of the remaining asymmetric ziggurat can be fitted into an n-brick.

Anticlimax. After all this struggle, we see our result writ large and clear in Pascal's Triangle (with $r = 2$) as the formula

$$\sum_{k=1}^{n} \binom{k}{r} = \binom{n+1}{r+1}\,, \tag{$\ddagger$}$$

which comes with its own cute induction proof, to wit:

$$\sum_{k=1}^{n-1} \binom{k}{r} + \binom{n}{r} = \binom{n}{r+1} + \binom{n}{r} = \binom{n+1}{r+1}\,.$$

For $r = 3$, it ultimately yields $n(n+1)/2 + (n-1)n(n+1) + (n-2)(n-1)n(n+1)/4$ as the sum of the first n cubes. However, this much less elegant than the direct induction proof that this sum equals $(1 + 2 + \cdots + n)^2$.