July 9, 2000.

Tangents of Angles Around a Polygon.

Problem #1 in *Vector* of Spring 2000: Prove that, for angles A,B,C in any triangle $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

Solution: Expand the equation $\sin(A + B + C) = 0$ by the addition laws, and then divide by $\cos A \cos B \cos C$.

To generalise this, remember that the source of the addition laws is the complex exponential $\cos A + i \sin A$, or — more modestly — the equation $R(A + B) = R(A)R(B)$ for the standard rotation matrices. Thus, $\sin(A_1 + \cdots + A_n)$ equals the imaginary part of the product $(\cos A_1 + i \sin A_1) \cdots (\cos A_n + i \sin A_n)$ for any sequence of angles. If these are the (exterior or interior) angles of a polygon, this quantity vanishes, and so does the imaginary part of $(1 + i \tan A_1) \cdots (1 + i \tan A_n)$, division by $\cos A_k$ being generically allowed. Therefore

$$\sum_{k\equiv 1 \bmod 4} \sigma_k(\tan A_1, \cdots, \tan A_n) \quad = \sum_{k\equiv 3 \bmod 4} \sigma_k(\tan A_1, \cdots, \tan A_n)\,,$$

where σ_k denotes the k-th elementary symmetric function.

3adic Weights.

Problem #2 *ibidem*: How many weights do you need to measure all values from 1 to 40 grammes on a simple two-pan pair of scales?

Solution: You can do it with weights of 1, 3, 9, and 27 grammes. Any weight can be in one of three states: "+" (left pan), "−" (right pan), and "*o*" (off). With n weights we therefore obtain at most 3^n configurations. Hence we need at least 4 weights. Since "*oooo*" (zero) can be omitted, and since all other values are duplicated by interchanging left and right, we actually get at most $(3^n - 1)/2 = 40$ different ones. But how do we get them all?

Any integer can be written in base 3 as $a_0 + a_1 3 + a_2 9 + a_3 27 + \ldots$, where the "digits" a_k are taken from a complete remainder system modulo 3. Normally we take $\{0, 1, 2\}$, but today we'll use $\{0, 1, -1\}$ written as $\{o, +, -\}$. To see that this is legal, we can either reconsider how the place-value system works using arbitrary remainders, or we can start with the standard representation using $\{0, 1, 2\}$ and replace every 2 by $3 - 1$, bumping up the next higher digit. Here are the numbers from 1 to 40 in the new representation:
$ooo+$, $oo+-$, $oo+o$, $oo++$, $o+--$, $o+-o$, $o+-+$, $o+o-$, $o+oo$, $o+o+$, $o++-$, $o++o$, $o+++$, $+---$, $+--o$, $+--+$, $+-o-$, $+-oo$, $+-o+$, $+-+-$, $+-+o$, $+-++$, $+o--$, $+o-o$, $+o-+$, $+oo-$, $+ooo$, $+oo+$, $+o+-$, $+o+o$, $+o++$, $++--$, $++-o$, $++-+$, $++o-$, $++oo$, $++o+$, $+++-$, $+++o$, $++++$.

If you want to feel young again, try multiplying two of these.

For a more systematic motivation of base 3, note that we need weights such that $(w_0 + w_1 + \ldots + w_k) + 1$ can realised by using the next higher weight, hence equals $w_{k+1} - (w_0 + w_1 + \ldots + w_k)$. Therefore $w_{k+1} = 2(w_0 + w_1 + \ldots + w_k) + 1$, which is $2(w_0 + w_1 + \ldots + w_{k-1}) + 1 + 2w_k = w_k + 2w_k = 3w_k$.