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Fool's Gold.

Some time ago, I saw an article which purported to give the following example of the Golden Section.

In the x, y-plane, consider three unit circles centered on the x-axis at $x = 1, 3, 5$. Let C be the point of contact between the upper tangent line from the origin to the third circle. This tangent line cuts the second circle in two points A and B. It is alleged that the collinear triple A, B, C is golden. Well, it ain't ...

Note that the ratios of distances between points on a line through the origin are equal to the corresponding ratios between differences of x-coordinates. We'll show that all points of intersection of yon line with our three circles have coordinates in the smallest field containing $\sigma = \sqrt{6}$. This includes the point $D \neq O$ on the first circle.

For starters, note that the distance OC equals $\sqrt{24} = 2\sigma$ by Pythagoras. Hence that tangent line has slope $1/2\sigma$ (just look at OMC standing on the base OC). Substituting $y = x/2\sigma$ in the equation $(x - t)^2 + y^2 = 1$ of a unit circle centered at $x = t$, yields the quadratic equation

$$\Big(1 + \frac{1}{24}\Big)\, x^2 + 2tx + t^2 - 1 = 0\,,$$

whose solutions work out to

$$x = \frac{12}{25}\left(2t \pm \frac{\sqrt{25 - t^2}}{\sigma}\right).$$

For $t = 1, 2, 3$, this gives $12/25$ multiplied by

$$2 \pm 2\,, \quad 6 \pm 4/\sigma\,, \quad 10\,,$$

respectively. Closest to golden is the ratio $BD : CD$, which yields $(3 + \sigma)/9$ or about 0.605. Too bad ... but proclaimed as gospel somewhere on the World Wide Web.