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Polyhedra and the Golden Mean.

On the unit sphere in 3-space imagine 3 unit circles, one around each of the coordinate axes. Since this text has no diagrams, let us describe them in geographic terms: the equator circles around the z-axis, which points upward; the 0-meridian (Greenwich) around the y-axis, which goes off to the right; and the 90°-meridian around the x-axis, which comes straight at you. In this setting consider the following 3 quadruples of points:

$$(0, \pm u, \pm v)\,, \quad (\pm v, 0, \pm u)\,, \quad (\pm u, \pm v, 0)\,, \tag{1}$$

with $u^2 + v^2 = 1$ and $u > v \geq 0$. If $v > 0$, each of them defines a $2u \times 2v$ rectangle in the appropriate coordinate plane.

To get oriented, let us begin by taking $v = 1/2 = \sin 30°$, and look at the six points in the closed northern hemisphere ($z \geq 0$). Then $A = (0, -u, v)$ is at New Orleans and $B = (u, -v, 0)$ just off the northeastern corner of Brazil, near the prison islands of Fernando Noronha. We'll find $C = (v, 0, u)$ near the Shetland Islands, $D = (u, v, 0)$ in Central Africa (Ruanda?), and $E = (0, u, v)$ in Tibet near Mount Everest. The sixth point $F = (-v, 0, u)$ is somewhere in the Aleutian archipelago, not visible from where we are looking (high above Bordeaux). The other 5 points A, B, C, D, E form a kind of **W** whose central peak is unusally tall.

The triangles ABC and CDE are equilateral. To see this, imagine v going to zero (and u going to 1): at the end, these triangles morph into the faces of a regular octahedron. As v increases again, their sides shrink in perfect unison. In total there are 8 such (regular) "trigons", all congruent, one in each octant. Their centres are always on the space-diagonals — which go through the points $(\pm w, \pm w, \pm w)$, with $w = 1/\sqrt{3}$, on the sphere. These centres (vertices of a cube) will a pl ay a role later on.

For the time being, we forget them and concentrate on triangles like BCD, which we shall call "darts" for short. Each of our twelve original points is the tip of exactly one dart. Each of the twelve darts is isosceles, sharing its legs with the aformentioned trigons, and sporting an acute vertex angle ($= 2\arctan(v/u)$) because $u > v$. Altogether we have a polyhedron with 20 faces: twelve darts plus eight regular trigons skewered on the space-diagonals.

Icosahedron.

If the darts were equilateral as well, we would have a *regular icosahedron*. To make this happen we would need to choose u, v in such a way that $|BD| = |DC|$, in other words: $4v^2 = |(v, 0, u) - (u, v, 0)|^2 = (u - v)^2 + u^2 + v^2 = 2u^2 + 2v^2 - 2uv$. Subtracting $2v^2$ and dividing by $2uv$, we change this to

$$\frac{v}{u} = \frac{u}{v} - 1\,, \tag{2}$$

the signature of the Golden Mean.

Conclusion: *the polygon with vertices as described in* (1) *is a regular icosahedron if and only if v/u is the Golden Mean.*

Dodecahedron.

Now let us get the centres $P = (w, -w, w)$ and $Q = (w, w, w)$ into the act. In the geographical model, P is about midway between the southern tip of Greenland and the northern coast of Brazil, while Q lies in southern Iraq.

We shall try to make $|BP| = |PC| = |CQ| = |QD| = |DB|$, and symmetry allows us to consider just the last of these equations. Now, $|DB|^2 = |QD|^2$, translates into $4v^2 = u^2 + v^2 - 2(u+v)w + 3w^2$ or, using $3w^2 = u^2 + v^2$, into $4v^2 = 2u^2 + 2v^2 - 2(u+v)w$ or even better, $u^2 - v^2 = (u+v)w$.

Thence it is easy to obtain

$$u - v = w \quad \Longrightarrow \quad uv = \frac{1}{3} \quad \Longrightarrow \quad \frac{u}{v} + \frac{v}{u} = 3\,, \tag{3}$$

the second equation by squaring the first (again using $3w^2 = 1 = u^2 + v^2$), and the third by dividing $u^2 + v^2 = 1$ by the second.

The last of these equations identifies v/u as the square of the Golden Mean (to see this, just square $\sqrt{5} - 1$ and $\sqrt{5} + 1$). Since this is the *only* way of making $|DB| = |QD|$, our new gizmo *must* be the regular dodecahedron (in other words, we need not check that B, P, C, Q, D are coplanar.

Conclusion: *the 12 points described in* (1) *plus the 8 points* $(\pm w, \pm w, \pm w)$ *are the vertices of a regular dodecahedron if and only if v/u is the Square of the Golden Mean.*

Edges and Inspheres.

Putting $q = v/u$, and denoting the Golden Mean by μ, we have seen that our construct is a regular icosahedron or dodecahedron depending on whether $q = \mu$ or $q = \mu^2$. Since $v = qu$, we have $u^2(1 + q^2) = 1$ and $v^2(1 + q^2) = q^2$. Since the base of each dart measures $2v$, *the edge-length is*

$$l = 2q(1 + q^2)^{-1/2} \tag{3}$$

in both cases. For the actual computing, remember that $\mu^2 = 1 - \mu$ and $\mu^4 = 2 - 3\mu$.

The radius r of the *insphere* can be computed as the distance from the origin to the midline of a dart. For BCD, the midline shows up in the xz-plane as the line segment connecting $(v, u) = C$ with $(u, 0)$, the midpoint of BD. The equation of that line being $x + (1 - v/u)z = u$, its z-intercept is $u/\sqrt{(1-q)}$, and its distance from the origin is

$$r = u\big(1 + (1-q)^2\big)^{-1/2} = \big((1 + q^2)(1 + (1-q)^2\big)^{-1/2} \tag{4}$$

(in a right triangle with sides a and b, a double computation of area yields $ab = r\sqrt{a^2 + b^2}$, where r is the altitude on the hypotenuse). The circle with radius r around the origin is therefore tangent to all four darts whose midlines lie in the xz-plane.

As $q = \mu$ implies $(1-q)^2 = q^4$ while $q = \mu^2$ makes $(1-q)^2 = q^2$, *the value of r is the same in both cases, namely:*

$$r = \frac{1}{\sqrt{(1 + \mu^2)(1 + \mu^4)}}\,. \tag{5}$$