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Simpson meets Archimedes.

The heart of "Simpson's Rule" is a lemma which might well have been known to Archimedes. In modern terminology and notation, it says:

The area under the graph of a quadratic function $q(x)$ on the interval $-h \leq x \leq h$ equals

$$\frac{h}{3} \cdot \Big(q(-h) + 4\,q(0) + q(h) \Big) .$$

Indeed let a, b, c denote those three values of q. To have a fixed picture in mind, assume that they are non-negative, and that the graph is concave downward, i.e. $b > (a + c)/2$. Subtracting a linear function whose graph includes $(-h, a)$ and (h, c) diminishes the area in question by the trapezoidal $h \cdot (a+c)$, and produces another quadratic function $p(x)$, whose graph includes $(\pm h, 0)$ and $(0, m)$, where $m = b - (a+c)/2$. As Archimedes had shown, the region under this new graph makes up $2/3$ of the rectangle $-h \leq x \leq h$, $0 \leq y \leq m$. Its area is therefore $4mh/3 = (4b - 2a - 2c) \cdot h/3$. Adding this to the trapezoidal $(3a+3c) \cdot h/3$ yields the desired result.

If we connect the three points $(-h, a)$, $(0, b)$, (h, c) by two straight lines instead of a parabolic arc, the resulting area is

$$\frac{h}{2} \cdot \big(a + 2b + c \big) .$$

It seems like such a good idea: instead of doing numerical integration by a series of trapezoids, use vertical strips bounded by the x-axis and a parabolic arc. But if the number of subdivisions is large, it does not seem to yield a great advantage. At least, I have not yet found a function where it does.