The Journey Through Cauchyland.

by Klaus Hoechsmann

Department of Mathematics, University of British Columbia, Vancouver, Canada

Cauchyland is the site of various theorems, formulas, and inequalities whose hypotheses and conclusions can be so diverse that one is sometimes tempted to ask :"Would the real M. Cauchy please stand up ?" The following outline sketches three itineraries, the first two of which are essentially parallel, each pursuing the globalization of a single local condition.

Itinerary 1 skirts the northern edge of Cauchyland, just south of the Plain of Sheaves, with a beautiful view of distant Mt. Potential; its main theme is the extension of primitives. Farther south, itinerary 2 follows a main road best known by its German name of "Weierstrasse", which leads through some of the main points of analyticity and ends at Lac St. Laurent. The third itinerary is transversal to these two and, because of its zig-zags and changing road conditions , is not recommended to travellers without some previous familiarity. It starts at the small border town of Cauchyville-Riemannstadt in Conformal County and passes through the hamlets of Goursat-sur-Yvette and Taylorton before it crosses the Sierra Morera.

0. Introduction. Let $U \subset \mathbf{C}$ be connected and open, $f : U \to \mathbf{C}$ continuous. For a point $c \in U$ consider the following three conditions on f.

D_c : In some neighborhood of c, f has a (complex) derivative.

I_c : In some neighborhood of c, f has an anti-derivative (also called "primitive").

A_c : In some neighborhood of c, f is given by a convergent power-series in $(z - c)$.

In fact, these conditions are equivalent– as we shall see later (at first sight, the only semi-obvious implications are $A \Rightarrow I$ and $A \Rightarrow D$). Conditions I and A have non-trivial global counterparts, namely

$I(U)$: f has a primitive defined on all of U.

$A(U)$: f is given, on all of U , by a single power-series.

The study of these conditions is undertaken in three main directions.

1. Globalization of primitives; i.e. moving from I_c $(\forall c \in U)$ toward $I(U)$. This direction is known as "Cauchy's Theorem".

2. Globalization of power-series; i.e. moving from A_c $(\forall c \in U)$ toward $A(U)$. This pursuit goes by the handle of "Cauchy's Integral Formula ", or "Cauchy's Inequalities". It requires some intermediate results from (1).

3. Connection with differentiability ; i.e. establishing the equivalence of D_c, I_c, A_c. This is the trickiest part and requires prior acquaintance with (1) and (2). In this game the names Goursat and Morera (and even Riemann) are commonly cited along with of course, Cauchy.

The main tool in these investigations is the *integral of f along a path γ* . Unfortunately for the expositor, there are *two* notions of such an integral, which for the purpose of this write-up will be distinguished by the terms "soft" and "hard".

The *soft* integral $\langle \gamma, f \rangle$ is defined for arbitrary (continuous) $\gamma : [0,1] \to U$ and for f satisfying I_c at all $c \in U$. It is the difference between the values at $\gamma(1)$ and $\gamma(0)$ of local primitives of f which can be linked by a consistent chain of local primitives along γ.

The *hard* integral $\int_\gamma f(z)dz$ is defined for arbitrary (continuous) f and for γ piece-wise smooth, so as to allow pulling back $f(z)dz$ to $f(\gamma(t))\gamma'(t)dt$ on the unit interval.

Of course

$$\langle \gamma, f \rangle = \int_\gamma f(z)dz$$

whenever both are defined. However, it would be very awkward to work out this theory on the intersection of their respective domains of definition: γ limited to "good" paths and f restricted to functions with local primitives. Note for instance, that condition I_c is not closed under multiplication and that "good" paths cannot be freely deformed.

1. Globalization of primitives. One of the main differences between calculus in **R** and in **C** comes from the diversity of shapes available for open connected subsets. (Remember that, in **R**, such a set would have to be a dull old interval.) Hence it is not surprising to find topological obstructions to one of the most basic problems of calculus : the construction of anti- derivatives.

Assume now that f satisfies I_c for every $c \in U$. The trip from this local condition toward $I(U)$ begins with a reformulation (1.1) of the latter and finishes with the two closely related results (1.2) and (1.2*).

(1.1) *LEMMA* : f satisfies $I(U)$ iff its integral is zero along every closed admissible path in U.

(1.2) *THEOREM* : f satisfies $I(U)$ if U is simply connected.

(1.2*) *THEOREM* : $\langle \gamma, f \rangle = 0$, if γ is homotopically (or homologically) trivial in U.

Intuitively, (1.2*) is fairly obvious, as long as one imagines γ to be a simple closed curve, say, the boundary of a blob $B \subset U$. Since the existence of local primitves makes "short" integrals depend only on end-points, γ may be replaced by a polygonal path, say, with horizontal and vertical segments. Paving B by tiny rectangles R_i and writing the integral over γ as the sum of integrals over the ∂R_i, we are reduced to admitting that each of the latter vanishes, provided (and we have!) that every R_i lies in the domain of a local anti-derivative.

The entire journey can be made with either the hard or the soft integral. Personally, I prefer the latter for its simplicity and elegance. One need only apply Lebesgue's Covering Lemma to get subintervals of $[0,1]$ into discs which carry local primitives; then the ordering on $[0,1]$ yields an inductive process of continuing the primitive along γ. The same idea applied to $[0,1] \times [0,1]$ shows that $\langle \gamma, f \rangle$ depends only on the homotopy class of γ.

To use the hard integral, one must first overcome some problems of definition–which either involves differential forms (Cartan) or rectifiability (Knopp), unless one restricts the "admissible" γ to something utterly simple , e.g. polygonal paths aligned with **R** and i**R** (Ahlfors). Since dodging the soft integral makes no sense except as a way of avoiding the inherent topology , one also faces the task of stating the main results without using homotopy. This is usually done by defining homology in terms of "winding numbers" (cf. below), creating the impression that all the secrets of plane topology somehow reside in the *log* function. Devotees of the hard integral might do well to leave aside these intricacies at first and to stop at useful particular cases of (1.2) and (1.2*) : U a disc or rectangle, γ the boundary of an eccentric annulus, etc. This can be done by the rectangular paving sketched above. However, if one looks closely at this procedure, one notices that it,too, requires something like Lebesgue covering.

So far, this whole business is an easy sheaf-theoretic manoeuver which applies equally well to, say, finding a (real-valued) potential function for a curl-free vector field.

We conclude with a result of a slightly different ilk, which could easily have been omitted, were it not for folks who would rather read paragraphs (2) and (3) than spend time in (1). For them we offer the following by-pass, the use of which will be demonstrated in (2).

(1.3) *SCHOLIUM* : Let U be the unit disc minus the origin,and suppose that $zf(z) \to 0$ as $z \to 0$. Then f satisfies $I(U)$.

The proof is first reduced to showing that $\langle \rho, f \rangle = 0$ for any rectangle ρ around 0. Subdividing the rectangle reduces us to small rectangles and a direct estimate, using the hypothesis. (cf. Ahlfors, Ch.III, Thm.5)

2. Globalization of power-series. Since the only distinction between $\mathbf{R}^2$ and $\mathbf{C}$ is algebraic, it is not surprising to find complex calculus dominated by an algebraic phenomenon : power-series. $f : U \to \mathbf{C}$ is called *analytic* in U, if it satisfies A_c for all $c \in U$. Notice that this encompasses just about all functions known to a beginner.

One easily sees that, modulo change of local parameters, such a function looks like $z \mapsto z^m$, hence is an open map, etc. A wealth of consequences flows from this , and one would be tempted to continue working with power-series except for the lack of information about their domains of convergence. This flaw is removed by means of the following construction. For *any* continuous $f : U \to \mathbf{C}$, and a loop γ in U, we form the *Cauchy Integral* $F_\gamma(s)$ of f with respect to γ by the formula

$$F_\gamma(s) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)dz}{(z-s)}$$

valid for s in the complement of γ. The motive for this formation will appear in the proofs below. For now, let us note that

(2.1) LEMMA : If c, s are interior points of a closed disc D contained in U, with $|s-c|$ less than the distance from c to ∂D,then

$$F_{\partial D}(s) = \sum_n \Big[\int_{\partial D} \frac{f(z)dz}{(z-c)^{n+1}} \Big] (s-c)^n$$

Indeed, integrating the geometric series expansion $(z-s)^{-1} = \sum_n (z-c)^{-n-1}(s-c)^n$, which is valid as long as c is closer to s than to z, we get the result.

(2.2) *THEOREM* : If f is analytic in U and s is an interior point of a closed disc D contained in U, then $f(s) = F_{\partial D}(s)$.

In all this the integrals have been of the "hard" variety, but they are unproblematic since $\gamma = \partial D$ is so simple. The deformability of integration contours has not been mentioned, but the proof of (2.2) does require a dip into (1). We give two proofs.

(2.2.a) *Proof* : Consider a function $g(z)$, for instance $f(z)/(z-s)$, which is given, in some neighborhood punctured at s, by a Laurent series $\sum_{-\infty}^{\infty} a_n (z-s)^n = a_{-1}(z-s)^{-1} + h(z)$. Then , for a small circle γ around s, $\langle \gamma, g \rangle = 2\pi i a_{-1}$, because $h(z)$ has a primitive defined on all of the punctured neighborhood. Hence $2\pi i f(s) = \langle \gamma, g \rangle = \langle \partial D, g \rangle$, by the mobility of the contour. (The transition from γ to ∂D only requires (1.2*) for an eccentric annulus).

(2.2.b) *Proof* : In view of the elementary formula

$$\int_{\partial D} \frac{f(z)}{(z-s)}\, dz = f(s) \int_{\partial D} \frac{1}{(z-s)}\, dz + \int_{\partial D} \frac{f(z)-f(s)}{(z-s)}\, dz$$

our task is twofold:
— show that $\int_{\partial D}(z-s)^{-1}dz = 2\pi i$,
— show that $\int_{\partial D}(f(z)-f(s))(z-s)^{-1}dz = 0$.

If you have no compunction about replacing ∂D by γ, a small circle around s, both are obvious : the first by direct computation, the second by the boundedness of the Newton quotient and the shrinkability of γ.

However, if you are dead set against contour deformation or if you just wish to keep (1) and (2) as tidily separate as possible, these tasks become more subtle. The second one can be referred directly to (1.3), but the first calls for an extra lemma.

(2.2.c) *Lemma* : For every loop γ not passing through s, the number

$$n(\gamma, s) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{(z-s)}$$

is an integer (called "winding number").

Prove this by using *Dlog* to show that $e^{h(t)}(\gamma(t)-s)$ is constant, where

$$h(t) = \int_{\gamma(0)}^{t} \gamma'(u)(\gamma(u)-s)du,$$

see?

Now, since $n(\gamma, s)$ is obviously continuous in s, it is constant on the interior of D, hence equal to 1. This accomplishes our task.

(2.3) *CONCLUSION* : The individual terms of the power-series in (2.1) give obvious formulas for the higher derivatives $F^{(n)}(c)$. Estimating these by means of $\max_{\partial D}|f(z)|$, one gets "Cauchy's Inequalities". Combining this with (2.2), we see that an analytic function f is given, in any disc, by a single power- series, whose n^{th} coefficient satisfies

$$|a_n| \le r^{-n}\max_{\partial D}|f(z)|$$

where r is the radius of D.

These facts could have been derived by applying the method of (2.2.a) to the higher terms of the local power series at the center of the disc, thus by-passing (2.1).

(2.4) *REMARKS* :

(i) Continuing along this line, it is not hard to see that an analytic f is given by a Laurent series in the interior of any closed *annulus* lying in U. Thus, in this paragraph, the annulus is the end of the line, much as the simply connected set was it in the last.

(ii) At this point it becomes possible to think of developping the rest of the theory along algebraic lines, using power-series. However, it is more convenient to keep using the power of Cauchy's formula: for instance, to see that uniform convergence on compacts preserves analyticity, that derivatives converge as the functions do, that bounded sets of analytic functions are equicontinuous , and the like.

(iii) Even if one is working with functions which are known to be basically analytic, the automatic analyticity (2.1) of the Cauchy Integral is useful for filling punctures or cracks in the domain . We give two examples (reminiscent of (1.3)) both situated in the unit disc U.

(2.4.1) If f is analytic and bounded in U minus the origin , it is analytic in U.

(2.4.2) If f is analytic and continuous in U minus the reals , it is analytic in U.

In both cases, one shows that Cauchy's formula – using a slightly smaller U' – is valid in U' minus the exceptional points.

3. Connection with differentiability. The time has come to prove the equivalence of our local conditions D, I, A. We shall discuss the implications $D \Rightarrow I$, $D \Rightarrow A$, $I \Rightarrow A$, in that order.

Let U be an open disc, $f : U \to \mathbf{C}$ continuous.

(3.1) *PROPOSITION (Goursat)* : If f has a derivative on U, then it has an anti-derivative on U.

One shows by successive quartering that

$$\int_{\partial R} f(z)dz = 0,$$

for any closed rectangle $R \subset U$... This is an *ab ovo* proof. If one could use the fact that $f = u + iv$ is real C^1, the proof would boil down to the following observation: the relations between the partials of u and v which say that the Frechet derivative of f is a complex multiplier (the "Cauchy-Riemann Equations") are the same as the equations which say that $f(z)dz$ is a closed differential.

(3.2) *PROPOSITION (Cauchy?)* : If f has a derivative on U, then it is analytic on U.

In fact, let $D \subset U$ be a closed disc, s an interior point of D. The proof $(2.2.b)$, that $f(s) = F_{\partial D}(s)$, requires only that $(f(z) - f(s))/(z - s)$, as a function of z, has local primitives everywhere in $D - \{s\}$; but this now follows from its differentiability.

(3.3) *PROPOSITION (Morera)* : If f has an anti- derivative on U, then it is analytic on U.

Indeed, by (3.2) , $f = g'$ implies the analyticity of g, hence of f.

(3.4) *REMARK* : A minimal itinerary through this theory might consist of a treatment of paragraph 2 with as much of paragraph 1 thrown in as suits the audience. In other words, one would concentrate on the exploitation of condition A, including a modicum of path deformation. The results of paragraph 3, though fascinating, are of little practical importance. Since the set of analytic functions is closed under algebraic operations and under uniform convergence on compacts, the implication $D \Rightarrow A$ is really never needed. It certainly is not the Main Theorem as which it is sometimes presented.

References.

Ahlfors,L.:*Complex Analysis,* McGraw-Hill,New York (1953).

Cartan,H.: *Théorie élémentaire des fonctions analytiques,* Hermann,Paris (1961)

Knopp, K.: *Theory of Functions,* Parts I and II, Dover, New York, (1945)