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Taylor. Andrew Adler has remarked to me that both forms of the remainder $R_n(x,a)$ in Taylor's formula $f(x) = T_n(x,a) + R_n(x,a)$ really come from the same simple fact about the a-derivative of the Taylor polynomial $T_n(x,a)$, to wit:

$$D_a\big(T_n(x,a)\big) = \frac{(x-a)^n}{n!}\, f^{(n+1)}(a)\,. \tag{1}$$

This formula itself follows immediately from the product rule for D_a applied to the summands $f^{(k)}(a)(x-a)^k/k!$ which make up $T_n(x,a)$.

$R_n(x,a) = f(x) - T_n(x,a)$ is therefore an antiderivative (with respect to a) of the negative of (1), and since $R_n(x,x)$ vanishes, we must have

$$R_n(x,x) = \int_a^x \frac{(x-t)^n}{n!}\, f^{(n+1)}(t)dt\,. \tag{2}$$

This is entirely formal and could be placed in the context of differential rings, with the integral read as an antiderivative. After all, it only says: for any $T(x,t)$ such that $T(x,x) = f(x)$, the difference $R(x,a) = f(x) - T(x,a)$ is the integral from a to x of $D_t T(x,t)$.

The second, mnemonically more attractive, form of the remainder follows from Rolle's Theorem applied to the auxiliary function $g(t) = (x-a)^{n+1}R_n(x,t) - R_n(x,a)(x-t)^{n+1}$. Since $g(x) = g(a) = 0$, there is a $z \neq x, a$ such that $g'(z) = 0$. By (1), the derivative $g'(t)$ becomes $\big[-(x-a)^{n+1}f^{(n+1)}(t)/n! + (n+1)R_n(x,a)\big](x-t)^n$. Substituting $t = z$ and dividing by $(x-z)^n$, we obtain

$$R_n(x,a) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(z)\,. \tag{3}$$

This whole business would not be worth revisiting, were it not for a strange habit of most (perhaps all?) calculus books. They derive (3) just as we did here and *then* get the really more immediate (2) via repeated integration by parts.

Euler. Suppose that a certain experiment has a 1 in n chance of going wrong, with n reasonably large. Then the probability of avoiding failure in a series of m experiments is obviously

$$\Big(1 - \frac{1}{n}\Big)^m \approx e^{-m/n} \approx (.37)^{-m/n}\,.$$

If you can improve your technique so as to give the single failure a chance of 1 in Cn, your run can therefore be increased to Cm without substantially affecting the probability of a flop. A strange case of pseudo-linearity!

In Paulos's book *Innumeracy*, this occurs without comment in the context of AIDS, where $n = 500$ and $m = 365$ — with $C = 10$ standing for condom.