The Leap Frog Game (Unfinished).

At the UBC Science Fair in April 1999, two girls won a prize with an exhibit about games including the famous Towers of Hanoi. A game which fascinated us (the judges) more was a simple track with 7 fields and 3 frogs at each end, xxx on the left and yyy on the right, facing each other across an empty field o. Thus the starting line-up was $xxxoyyy$, and the aim of the game was to change this to $yyyoxxx$ by a succession of permitted moves. These were the shifts $S_x : xo \mapsto ox$ and $S_y : oy \mapsto yo$ as well as the jumps $J_x : xyo \mapsto oyx$ and $J_y : oxy \mapsto yxo$.

Let us consider the general problem of changing $x^n oy^n$ to $y^n ox^n$. The trick is to produce first a "splicing" $(yx)^n o$ or $o(yx)^n$ and then reverse the procedure to make the y's agglutinate on the left. For $n = 1$, the 3-step pattern $xoy \mapsto oxy \mapsto yxo \mapsto yox$ is too simple to reveal anything. In the case $n = 2$, the splicing is done in these steps

$$x^2oy^2 \mapsto x(oxy)y \mapsto x(yxo)y \mapsto xyxyo \mapsto oyxyx\,, \qquad (*)$$

namely S_x followed by J_y and then S_y followed by J_x^2. This makes for a total of 3+2 elementary operations. The unsplicing is done by interchanging x and y and then going backwards on the first three arrows in reverse order (see?). Altogether the process takes 3+2+3 operations.

If we use the same splicing steps on the inner bracket of $x^3oy^3 = x(x^2oy^2)y$ we are left with $x(oyxyx)y = xo(yx)^2y$, whence we can complete the splicing

$$xo(yx)^2y \mapsto o(xy)^3 \mapsto (yx)^3o$$

using S_x followed by J_y^3. We begin the unsplicing by switching x and y and reversing the first of these arrows to yield

$$S_y : o(yx)^3 \mapsto yo(xy)^2x\,,$$

whence the unsplicing of $(*)$ — remember: only three arrows are reversed — makes $y(y^2ox^2)x$. Total number of elementary operations:

Fuck it, it's getting late. Notabene:

$$xo(yx)^{k-1}y \mapsto o(xy)^k \mapsto (yx)^ko \quad \text{or} \quad x(yx)^koy \mapsto (xy)^{k+1}o \mapsto o(yx)^{k+1}$$

via S_x followed by J_y^k, or S_y followed by J_x^{k+1} shows how the splicing progresses. It is always S_a followed by J_b^k, and the total number of elementary operations is $(1 + 2 + \cdots n)$ for the jumps plus n for the shifts.

On the way down (unsplicing), the last n-fold jump is not needed (x and y having been switched), so we get another $(1 + 2 + \cdots n)$ for total of $n^2 + 2n$.