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Mars for Simpletons.

Imagine the earth orbiting the sun in a circular orbit to the tune $E(\theta) = e^{i\theta}$, and another planet doing likewise with $P(\theta) = Re^{i\omega\theta}$. Of course, the orbit of mars is notoriously non-circular — but ignoring that fact, we would set $R = 1.52$ and $\omega = .53$ (for Venus, those values would be .72 and 1.61, respectively). The line of sight from earth to the planet is determined by the vector $X(\theta) = P(\theta) - E(\theta)$, which — interpreted as as point (i.e., with the earth as origin) — is clearly an epicycle.

Writing Cartesian coordinates for $X(\theta)$ and its velocity vector $\dot{X}(\theta)$, we get

$$X(\theta) = \big[\quad R\cos\omega\theta - \cos\theta, \quad R\sin\omega\theta - \sin\theta\big] \quad \text{and}$$
$$\dot{X}(\theta) = \big[-\omega R\sin\omega\theta + \sin\theta, \omega R\cos\omega\theta - \cos\theta\big]\,,$$

respectively. The planet is retrograde, when $\dot{X}(\theta)$ is on the "wrong" side of $X(\theta)$, in other words, when

$$\det\big(X(\theta), \dot{X}(\theta)\big) = (R^2\omega + 1) - R(\omega+1)\cos(\omega-1)\theta$$

is negative. This boils down to the inequality

$$\cos(\omega-1)\theta \quad > \quad \frac{R^2\omega+1}{R(\omega+1)} \quad = \quad 1 \quad + \quad \frac{(R-1)(R-T)}{R(1+T)}\,, \tag{$*$}$$

where $T = 1/\omega$ is the duration of the planet's year (in earth years). This is solvable if and only if $R-1$ and $R-T$ have opposite signs: outer planets must have long years, inner planets short years (compared to their radii) for retrogression to occur.

"Kepler's Third Law" $R^3 = T^2$, here an easy consequence of the gravitational inverse square and the old centrifugal story, turns the right hand side of $(*)$ into $\sqrt{R}/(R-\sqrt{R}+1)$. In a "solar" system, therefore, retrogression always does occur — but this may not happen in other systems of concentric motion, like earth-moon-sun.

The data for Mars quoted at the outset would yield $-35.5^\circ < \theta < 35.5^\circ$, i.e., roughly 72 days (for Venus it would be $-21.6^\circ < \theta < 21.6^\circ$, or roughly 43 days). Are these compatible with observation? Check it out!

The visibility of the planet in the night sky is another matter. It depends on the positivity of the scalar product $X(\theta) \bullet E(\theta) = R\cos(\omega-1)\theta - 1$, in other words, the inequality

$$\cos(\omega-1)\theta \quad > \quad \frac{1}{R}\,,$$

whose right hand side is smaller than that of $(*)$ if and only if $R > 1$. This means that retrograde Mars is always visible in the night sky, but Venus does much of her retrogression invisibly by day — which is not surprising if we remember that $\theta = 0$ means collinearity of sun, earth and planet.