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Multiple Vector Products.

Given two pairs V_1, V_2 and W_1, W_2 of vectors in 3-space, let us first derive the formula

$$(V_1 \times V_2) \bullet (W_1 \times W_2) = (V_1 \bullet W_1)(V_2 \bullet W_2) - (V_1 \bullet W_2)(V_2 \bullet W_1)\,. \tag{1}$$

Indeed, writing V_1, V_2 as rows and W_1, W_2 as columns, we have the square matrix product

$$\begin{bmatrix} V_1 \\ V_2 \\ X \end{bmatrix} \cdot \left[W_1^T, W_2^T, X^T\right] = \begin{bmatrix} V_1 \bullet W_1 & V_1 \bullet W_2 & V_1 \bullet X \\ V_2 \bullet W_1 & V_2 \bullet W_2 & V_2 \bullet X \\ 0 & 0 & X \bullet X \end{bmatrix} \tag{2}$$

for any X orthogonal to both W_1 and W_2. Remembering that the cross product $V_1 \times V_2$ is defined by

$$\det \begin{bmatrix} V_1 \\ V_2 \\ X \end{bmatrix} = (V_1 \times V_2) \bullet X\,, \tag{3}$$

we obtain

$$\big((V_1 \times V_2) \bullet X\big) \cdot \big((W_1 \times W_2) \bullet X\big) = \big((V_1 \bullet W_1)(V_2 \bullet W_2) - (V_1 \bullet W_2)(V_2 \bullet W_1)\big) \cdot \big(X \bullet X\big)\,,$$

by taking determinants on both sides of (2). Putting $X = W_1 \times W_2$, we finally get (1).