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The Right Musical Scale.

d'après Adrian Lewis.

The problem is to cut the octave into n equal pieces in such a way that the m-th step falls close to the dominant. In other words, find m and n such that

$$2^{\frac{m}{n}} \approx \frac{3}{2} \qquad \text{or} \qquad \frac{m}{n} \approx \log_2 3 - 1\,.$$

We know from Dirichlet *et al.* that the continued fraction expansion $(a_o, a_1, a_2, \ldots)$ yields optimal rational approximations of a real number $\alpha = \alpha_0$. Recall that a_n is the integral part of α_n, where $\alpha_n = (\alpha_{n-1} - a_{n-1})^{-1}$ is the inverse of the fractional part of α_{n-1}.

For $\alpha = \log_2 3 - 1$ we get the continued fraction $(0, 1, 1, 2, 2, 3, 1, \ldots)$, which gives the rational approximations

$$\frac{1}{2}\ ,\ \frac{3}{5}\ ,\ \frac{7}{12}\ ,\ \frac{24}{41}\ ,\ \frac{31}{53}\quad \ldots$$

The third of these is the basis of Western music. It is rumoured that the one with 31/53 was known 2000 years ago in Ancient China. Was it ever used?

Of course, a scale should not only get close to the dominant, but also to the subdominant as well as to the major third and sixth. The 53-step scale does very well: its 17-th, 22-nd, 31-st, and 39-th step yield 1.249, 1.333, 1.500, and 1.665, respectively, up to 3 decimal places. The 41-step scale would be only slightly worse for the third (1.246) and the sixth (1.661). But then, why cut it that fine? If twelve tones were good enough for Mozart, they are surely adequate for me.

Speaking of the devil: in one of Mozart's late quartets, the first three notes are the tonic, the octave, and the "tritone", i.e. the 6-th step of our scale – considered diabolic in medieval music. Since it corresponds to the square root of 2, it presumably would have made Pythogaras squirm – whence we can surmise an answer to Lewis's title question: *Would Pythogaras Have Liked Mozart?*