Nov. 27, 1999.

Stacking Slabs.

We are stacking a pile of 12 inch wooden slabs (say, 1-by-4-by-12) one on top of the other. Actually it would be more accurate to say that we stack them one *below* the other, because when we have put $(n-1)$ of them in place, we carefully lift the whole assembly and slip the n-th slab underneath it. This is done in order to keep increasing the total width of our stack without allowing it to tip over. We are working on a table top with a yardstick nailed along an edge — for future reference. Our slabs are aligned lengthwise with that yardstick.

The reason for building from the top down becomes clear as soon as we try to place the third slab. If the first two are laid so as to have a reach of 18 inches, no further extension is possible (without tipping) by putting another slab on top — but it is not hard to make three layers reach 24 inches by building downward (the way gravity pulls). Every time we slip a new slab under the assembly, we can recess it a bit from the edge of the previous one — so as to increase the reach — but these "over-hangs" will have to get shorter and shorter to counteract gravity. Eventually our construction will look like one half of an arch. What kind of an arch?

We choose some point on the yardstick as our "pivot" (it does not have to be the beginning of the stick). Measuring distance in feet and using our slabs as "unit weights", we get a total torque around the pivot of nx_n for a stack of n layers. The distance x_n defines a point at which the whole weight n would produce the same torque as the individual pieces in their respective positions taken together. It is a kind of average.

It is important that torques can be *added*, so that

$$nx_n = (n-1)x_{n-1} + t_n \tag{1}$$

where t_n is the torque caused by the n-th slab alone. If we place the front edge of the n-th slab at the distance d_n from the pivot, we get $t_n = d_n + 1/2$ (where the 1/2 represents the distance from edge to centre of a slab).

In order to prevent tipping, this n-th edge must be no more than x_{n-1} feet from the pivot. Therefore $d_n = x_{n-1} - s_n$, where s_n is a small (ideally zero) non-negative "safety margin". Altogether, we get $nx_n = nx_{n-1} + 1/2 - s_n$ or

$$x_n = x_{n-1} + \frac{1}{2n} - \frac{s_n}{n}\,, \tag{2}$$

where s_n is to be thought of as tending to zero. In the ideal case, all s_n actually *are* zero, and (choosing the pivot so that $x_1 = 1/2$) we get

$$x_n = \sum_{k=1}^{n} \frac{1}{2k} \tag{3}$$

for all $n > 0$, and this also represents the front edge of the $(n+1)$-st slab.