October 1997

Fractions of a quadrant.

1. In a Cartesian coordinate system on graph paper (with unit length = 10 cm) draw the semi-circle containing the points $I = (1, 0)$, $J = (0, 1)$, and $I' = (-1, 0)$. Locate $P = (.8, .6)$ and the mid-point M of the line segment IP. Compute the coordinates of M, as well as its distance from the origin $O = (0, 0)$. Now find the coordinates of the point $P^{1/2}$ where the line OM intersects the circle.
2. Repeat Exercise 1 for $P = (-.6, .8)$, again for $P = (8/17, 15/17)$, and then for $P = (1/\sqrt{2}, 1/\sqrt{2})$.
3. Repeat once more for an arbitrary $P = (u, v)$ on the semi-circle; in other words, find $P^{1/2} = (x, y)$ which bisects the angle POI. *Hint:* For slickest results, note that $(\frac{u+1}{2})^2 + (\frac{v}{2})^2 = \frac{u+1}{2}$ in this case.
4. The following table contains the coordinates of the points which define certain fractions of the first quadrant. Make sure you understand how to read it. Then use your procedure from Exercise 3 to extend it to the fractions $k/16$, where $k = 1, \ldots, 15$.

1/8	0.980785280	0.195090321	7/8
1/4	0.923879532	0.382683432	3/4
3/8	0.831469612	0.555570233	5/8
1/2	0.707106781	0.707106781	1/2

5. An idling engine makes one full revolution every 64 milliseconds, and its piston moves at a maximal speed of 10 metres per second. Using the table from Exercise 4, make a graph of the piston speed (vertically, 1 cm = 1 metre per second) as a function of time (horizontally 1 cm = 1 millisecond) during one quarter revolution (starting at 0).
   How does this graph give approximate values for trigonometric functions?
6. For every number $0 < s < 1$ the following BASIC-programme prints a string $b(s)$ of zeroes and ones. What does it represent? (Try it first with $s = 1/\sqrt{2}\,, 1/2\,, \sqrt{3}/2$).

```
1 m=0:u=0:v=1:input s:r=sqr(1-s*s)
2 m=m+1:u=sqr(1/2+u/2):v=v/(u+u):b=0
3 if s<v goto 5
4 x=r*u+s*v:s=s*u-r*v:r=x:b=1
5 print b;:  goto 2
```
   (or: `if m < 30 goto 2` in order to stop at 30 places).

7. The programme above can also be worked with calculator, pencil, and paper — especially well if lines 2 and 4 can be handled by two different players.
   (a) In this manner find the missing 6 places for each of the sequences

   $b(3/5)$ =??????0011011111110101110... und $b(15/17)$ =??????0000100110001111010...

   (b) Convert $b(3/5)$ and $b(15/17)$ into decimal arc-measures (as fractions of a quadrant) for the points $(4/5, 3/5)$ and $(8/17, 15/17)$ on the semi-circle. For convenience, you may wish to change them first into hexadecimal notation.
8. For every $P = (u, v)$ on the semi-circle, let $\Delta(u, v)$ denote the area of the triangle $(0, 0)$, (u, v), $(1, 0)$. Compute $\Delta(u, v)/\Delta(x, y)$, where $(x, y) = P^{1/2}$ as in Exercise 3. By what factor is the area of the regular 16-gon greater than that of the regular octagon?
9. The following programme is due to François Viète (ca. 1570). Go through its loop 10 times with a calculator. What does it approximate — and by what logic?

```
1 a=2:u=0:n=4
2 u=sqr(1/2+u/2):a=a/u:n=n+n
3 print n,a:  goto 2
```