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Computing Roots.

Consider a rectangle with base b, height h, and area $a = bh$. If $b \neq h$, we can create a new, more *square-like* rectangle with the same area a, by *averaging* base and height, that is, by taking

$$b_* = \frac{b+h}{2} \qquad \text{and} \qquad h_* = \frac{a}{b_*}$$

as our new base and height. This process can be repeated until the rectangle is as square-like as we wish. The following list shows how this works for $a = 9$ and $b = 2$. The results are given both as common fractions and as decimals. We start with $b_1 = 2/1 = 2.0$ and $h_1 = 9/2 = 4.5$. Then:

$b_2 = \frac{13}{4} = 3.25$ $\qquad$ $h_2 = \frac{36}{13} = 2.769230769230769231$

$b_3 = \frac{313}{104} = 3.009615384615384615$ $\qquad$ $h_3 = \frac{936}{313} = 2.990415335463258786$

$b_4 = \frac{195313}{65104} = 3.000015360039321701$ $\qquad$ $h_4 = \frac{585936}{195313} = 2.999984640039321499$

$b_5 = \frac{76293945313}{25431315104} = 3.000000000039321600$ $\qquad$ $h_5 = \frac{228881835936}{76293945313} = 2.999999999960678400$

Good — but not perfect. Of course, $b = h = 3$ would fit perfectly, but our algorithm does not know that. Let us see what it does with $a = 10$, $b = 3$. We start with $b_1 = 3/1 = 3.0$ and $h_1 = 10/3 = 3.333333333333333333$. Then:

$b_2 = \frac{19}{6} = 3.166666666666666666$ $\qquad$ $h_2 = \frac{60}{19} = 3.157894736842105263$

$b_3 = \frac{721}{228} = 3.162280701754385964$ $\qquad$ $h_3 = \frac{2280}{721} = 3.162274618585298196$

$b_4 = \frac{1039681}{328776} = 3.162277660169842080$ $\qquad$ $h_4 = \frac{3287760}{1039681} = 3.162277660166916583$

$\frac{2161873163521}{683644320912} = 3.162277660168379331$ $\qquad$ $\frac{6836443209120}{2161873163521} = 3.162277660168379331$

Better — at least on the decimal side. The common fractions still look different, but are they? Is there a perfect square root of 10 (as 3 is the square root of 9)? Of course it could not be a whole number, but it might be a common fraction.

Exercises.

1. Using the same method, approximate $\sqrt{10}$ (a) by decimals, (b) by common fractions, starting with $b_1 = 5/2 = 2.5$. Use a calculator for the computations.

2. Let $d_k = |b_k - h_k|$ be the diffference between base and height at the k-th turn of this algorithm. What can you say about the size of d_{k+1}? (Hint: visualize b_k, h_k, and b_{k+1} on the number line.)

3. Suppose there were a fraction n/m, with n and m whole numbers, such that $(n/m)^2 = 10$. Then the whole numbers n^2 and $10m^2$ would be equal (see?). *This is impossible because* … Complete the last sentence with an unassailable reason.

4. Devise a method for approximating cube roots. Starting with a square prism of base $b \times b$ and height h, create a more cube-like object (with the same volume) by averaging: $b_* = \frac{1}{3}(b + b + h)$, and so on. Try this out for computing the cube root of 9, using decimals.