Klaus Hoechsmann, March 3, 1998

Bovine Equations.

These are $n \times n$ systems of linear equations of the form

$$x_i = a_i x_{i+1} + b_i \,, \tag{1}$$

with the indices i to be read modulo n. To find x_k, we simply substitute:

$$x_k = a_k(a_{k+1}x_{k+2} + b_{k+1}) + b_k = a_k\big(a_{k+1}(a_{k+2}x_{k+3} + b_{k+2}) + b_{k+1}\big) + b_k$$

and so on through $n-1$ steps, until we have $(1 - a_1 \cdots a_n)x_k =$

$$b_k + a_k b_{k+1} + a_k a_{k+1} b_{k+2} + \cdots + (a_k \cdots a_{k+n-2}) b_{k+n-1} \,. \tag{2}$$

Of course $d = 1 - a_1 \cdots a_n$ is the determinant of the system, and had better be non-zero. The term "bovine" (mine) is to remind us of the famous cattle problem of Archimedes (cf. the article by Friedrich Schwarz in DMV-Mitteilungen 2/97, pp. 13 – 18).

Archimedes uses two such systems: for the 4 kinds of steers in the herd of Helios, he let $b_1 = b_2 = b_3 = b_4$ denote the number of checkered steers, while x_1, x_2, and x_3 stood for the numbers of white, black, and brown ones. Hence he has $n = 3$, and he takes $a_1 = 5/6$, $a_2 = 9/20$, $a_3 = 13/42$, thus making $d = 99/112$. By (2), we obtain the solution

$$(x_1, x_2, x_3) \;=\; d^{-1}(1 + a_2 + a_1 a_2,\; 1 + a_2 + a_2 a_3,\; 1 + a_3 + a_3 a_1)\, b \,, \tag{3}$$

where b is the common value of the b_i. Since this is one of the unknowns, we might as well relabel it as x_4. A fundamental rational solution of the 4×4 homogeneous system so obtained can be read off from (3) by setting $x_4 = b = 1$. Its least common denominator turns out to be 81×11. Multiplying through by this, we get the smallest integral solution $(x_1, x_2, x_3, x_4) = (2226, 1602, 1580, 891)$. The actual numbers of steers are therefore

$$(\xi_1, \xi_2, \xi_3, \xi_4) = (2226c, 1602c, 1580c, 891c)$$

for some positive integer c.

The second system concerns the corresponding cows, but also refers to the numbers of steers. In terms of (1), Archimedes here takes $n = 4$ and $b_i = a_i \xi_i$, with $a_1 = 7/12$, $a_2 = 9/20$, $a_3 = 11/30$, and $a_4 = 13/42$, which yields the determinant $4657/4800$. The integrality condition drives c up to 4657 (a prime). What pushes the numbers over the top, however, are two further constraints: $\xi_1 + \xi_2$ should be a square number (that's not too bad) and $\xi_3 + \xi_4$ a triangular one (there's the kicker!).