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The Doughnut Problem.

Given $z = xy - x - y$, the identity

$$ax - by = 1 \tag{1}$$

quite formally entails

$$z + 1 = (a-1)x + (x-b-1)y \quad \text{and} \quad z - 1 = (y-a-1)x + (b-1)y\,. \tag{2}$$

This observation can be used to prove the following theorem.

Theorem. If x and y are relatively prime integers greater than 1, then z is the greatest integer which is not of the form

$$mx + ny \quad \text{with} \quad m \geq 0\,, \quad n \geq 0\,. \tag{3}$$

Proof. The set of pairs (a, b) of integers satisfying (1) is a double arithmetic series $(a_k, b_k) = (a_0 - ky, b_0 - kx)$ with k running through all integers, and (x_0, y_0) representing some fixed initial pair. The smallness of the right hand side of (1) guarantees that a and b are always of the same sign.

Let us choose $a = a_0$ to be minimal positive. Then

$$a_{-1} = a - y < 0 < a = a_0 \quad \text{and} \quad b_{-1} = b - x < 0 < b = b_0\,, \tag{4}$$

showing that b is minimal positive as well. More importantly, both $y - a$ and $x - b$ are positive, and hence all the bracketted coefficients in (2) are non-negative. Thus both $z - 1$ and $z + 1$ have the form (3).

But z itself does not have this form. If it did, we could subtract it from the first part of (2) and obtain

$$1 = (a - 1 - m)\,x + (\cdots)\,y\,,$$

whence $m + 1 = ky$ for suitable $k > 0$. Now $k = 1$ is impossible: it would make $m = y - 1$ and $ny = z - mx = -y$. On the other hand, $m = ky - 1$ with $k > 1$ would make $m > y$ and $mx + ny > yx > z$.

It remains to show that every $w > z$ does have the form (3) — which is true for $w = z + 1$ by the first equation of (2). The rest follows by induction: $w = mx + ny$ with $m, n \geq 0$ implies $w + 1 =$

$$(m + a)x + (n - b)y = (m + a - y)x + (n - b + x)y\,, \tag{5}$$

with the bracketted quantities non-negative in at least one of these forms. In fact, if both $n \leq b - 1$ and $m \leq y - a - 1$, the second equation in (2) would make $w \leq z - 1$. Therefore, if the left hand side of (5) is in trouble because $b < n$, the right hand side can take over with $m \geq y - a$, and hence $m + a - y \geq 0$. The other coefficient $n + x - b$ is automatically non-negative by (4). $\square$