Klaus Hoechsmann, February 24, 2008.

Niven's proof of the irrationality of π.

In August 2007, Wikipedia contained a version of Niven's proof (cf. next page) which has since been replaced because it was flawed. It was, however, quite elegant and the point of this commentary is to salvage it. The crux of Niven's proof is the use of the supposed rationality of π to invent a *small positive* function f such that the integral of $f(x)\sin(x)dx$ over $[0,\pi]$ is an integer. In this exposition we avoid the manipulation of integrals and directly exhibit a function G with

$$G' = f\cdot\sin \qquad \text{and} \qquad G(\pi) = -G(0) \in \mathbf{Z}\,. \tag{1}$$

The positivity of its derivative forces $G(x)$ to be monotone increasing from $G(0)$ to $G(\pi)$, and since the difference of these heights is $2G(\pi) \geq 2$ and $\pi < 4$, the whole scenario is impossible if $0 < f(x) < 1/2$.

1. The key intermediate step is to construct an F such that

$$F'' + F = f \qquad \text{and} \qquad F(\pi) = F(0) \in \mathbf{Z}\,. \tag{2}$$

Then

$$(F'\cdot g - F\cdot g')' = F''\cdot g - F\cdot g'' = (F''+F)\cdot g = f\cdot g\,, \tag{3}$$

whenever $g'' = -g$, for instance $g = \sin$, and we can set $G(x) = F'(x)\sin x - F(x)\cos x$.

2. If f is a polynomial, its derivatives will eventually vanish, and we get $D^2F + F = f$ by the geometric series for $1/(1+D^2)$ applied to f, i.e.,

$$F(x) = f(x) + \cdots + (-1)^j f^{(2j)}(x) + \cdots + (-1)^n f^{(2n)}(x)\,, \tag{4}$$

We are still left to worry about the size of f, the integrality of $F(0)$ and $F(\pi)$ as well as their equality. For the latter, a symmetry $f(x) = f(\pi - x)$ would be nice.

3. Suppose $\pi = a/b$ for some positive integers a and b. For any n we can define

$$f(x) = \frac{x^n(a-bx)^n}{n!}\,. \tag{5}$$

The higher even derivatives of f satisfy the identities

$$f^{(2j)}(x) = f^{(2j)}(\pi - x)\,, \tag{6}$$

obviously for $j = 0$, and further by twofold application of $f'(x) = -f'(\pi - x)$ and the like. Clearly $f^{(2j)}(0) = f^{(2j)}(\pi) = 0$ for $2j \leq n$, while beyond that point, the polynomial $f^{(2j)}(x)$ has integer coefficients and vanishes completely for $j \geq n$. For F as defined in (4), this means that $F(\pi) = F(0)$ is an integer.

4. Since $f(x)$ satifies

$$0 \leq f(x) \leq \frac{\pi^n a^n}{n!} \quad \text{for} \quad 0 \leq x \leq \pi < 4\,, \tag{7}$$

it would suffice to choose n so as to make $n! > 2\cdot(4a)^n$, to wind up with $0 < f(x) < 1/2$ over $(0,\pi)$.

Copied from Wikipedia, August 19, 2007.

Niven's proof

Like all proofs of irrationality, the argument proceeds by reductio ad absurdum. Suppose π is rational, i.e. $\pi = a/b$ for some integers a and b, which may be taken without loss of generality to be positive.

Given any positive integer n we can define functions f and F as follows:

$$f(x) = \frac{x^n(a-bx)^n}{n!},$$

$$F(x) = f(x) + \cdots + (-1)^j f^{(2j)}(x) + \cdots + (-1)^n f^{(2n)}(x).$$

[KH: This is the expansion of $1/(1+D^2)$ applied to f, whence $(1+D^2)F = f$, cf. below.] Then f is a polynomial function each of whose coefficients is $1/n!$ times an integer. It satisfies the identity

$$f(x) = f(\pi - x)$$

and the inequality

$$0 \leq f(x) \leq \frac{\pi^n a^n}{n!} \quad \text{for} \quad 0 \leq x \leq \pi. \qquad (*)$$

Observe that for $0 < j < n$, we have

$$f^{(2j)}(0) = f^{(2j)}(\pi) = 0. \qquad \text{WRONG!}$$

For $j = n$, $f^{(2j)}(0)$ and $f^{(2j)}(\pi)$ are integers - negative integers for odd n, and positive integers for even n. Consequently $F(0)$ and $F(\pi)$ are positive integers. Next, observe that

$$F + F'' = f,$$

and hence

$$(F' \cdot \sin - F \cdot \cos)' = f \cdot \sin .$$

It follows that

$$\int_0^\pi f(x) \sin(x)\, dx$$

is a positive integer. But by the inequality (*), the integral approaches 0 as n approaches infinity, and that is impossible for a sequence of positive integers.