Klaus Hoechsmann, Aug. 20, 1995

Informal Responses to Elementary Problems About Chance.

Consider the following nine problems. The question is to what extent each of them calls for — or, at least, is likely to elicit — a response which shows the respondent's background in mathematical probability. Apart from using jargon (which future teachers will tend to avoid) there seems to be no way of *explicitly* bringing such background to bear on the first six. For them, the informal answer is not only adequate but also conclusive; for the others, it goes a long way toward a solution — as will be demonstrated below. The responses, shown in italics, are those of an imaginary individual who has had no math beyond grade 5, but whose thinking is practical, incisive, and down-to-earth.

1. A fair coin is flipped five times, each time landing with heads up. What is the most likely outcome if the coin is flipped a sixth time?

If it is really fair*, it would come down 50-50 heads or tails any time you toss it. It has no memory of what happened before. So, the chances on the sixth flip are still 50-50.*

2. On an inclined plane with a system of channels, as in the diagram on the right, a marble is rolling down from point A in the direction of the arrow. Where do you think it would come out at the bottom?

On its way down, the marble first meets one fork in its path and then another. Both times it either turns right or left, presumably with equal likelihood. Its exit depends on which of the 4 patterns RR, RL, LR, or LL has occurred. As in flipping 2 coins, they are equally likely.

3. Two people disagree about selecting numbers for the 6/49 lottery. One of them prefers consecutive numbers like 1,2,3,4,5,6, the other thinks that the chance of winning the lottery is greater with a random sequence. How would you advise them?

The numbers are written on 49 balls kept in a tumbler, but it would be the same if they were written on chips in a bowl. At drawing time, six of them are allowed to plop out. The numbers on them do not affect this at all. Any specific sequence is as likely or unlikely as any other.

4. In the diagram on the right, OA is a spinner. Each of four players picks two numbers from 1 to 8, and if the spinner stops on one of them, the person wins. Which numbers would you pick to have the best chance of winning?

Let's say OA is the minute-hand of a watch, and the betting is about where it will stop when the battery runs out. The dial has been divided into 8 sectors, each covering seven and a half minutes. Whichever two sectors you choose — you are occupying fifteen minutes out of sixty. Your chance of winning does not depend on your choice: it is always 25%.

5. In the diagram on the right, OA is a spinner. On which number is it most likely to stop?

As in the last question, imagine a minute-hand. This time there are five unequal sectors, and most of the dial has been broken off, except for the rectangular piece shown in the diagram. One of the sectors, namely #3, covers fifteen minutes, the others visibly cover less. So, #3 has the greatest chance of being selected.

6. A thumb tack falls to the floor. When it comes to rest, is it more likely to be pointing straight up or lying on its side pointing downward?

This depends on the design of the thumb tack. I could make you one (with a heavy rounded head) which would always stand up straight no matter how it landed — or one which would always lie down like a roofing nail. If you give me your tack, I would keep tossing it, until I saw a clear tendency one way or the other. If this doesn't happen, I'd bet 50-50.

In the last three questions, an informal response is no longer as conclusive as in the first six. Nevertheless, it is interesting to see to what extent it can size up and clarify the situation.

7. Which is more likely, that you win the 6/49 jackpot or that you die in a car accident?

If there are 2,000,000 people in this city, and only one of them is destined to die in a car accident (dream on!), my chance of being hit is at least 1 in 2,000,000 — barring special dispensation. I've heard that the chance of hitting the 6/49 jackpot is only about 1 in 14,000,000 — which would be much smaller. (Given more time, I'd try to figure out where they got the 14,000,000.)

8. A bank has two teller windows, each serving one customer per minute. Customers arrive at a rate of between 1 and 6 per minute. If the average waiting time is to be no more than 3 minutes, should the number of tellers be increased or decreased?

If the average number of incoming customers is more than 2 per minute, the 2 tellers won't be able to prevent an unlimited build-up of clients, whose waiting time will grow with their number. To limit waiting time, the number of tellers must at least match the average number of customers arriving per minute. To find that number, I'd need more data.

9. In the following game, 3 dice are rolled by the "dealer", after the "player" has staked a certain amount of money on a number from 1 to 6. For every dollar the player has staked, the dealer pays \$1 if that number comes up once, \$2 if it comes up twice, and \$3 if it comes up three times. If it does not show on any of the three dice, the player loses the stake, but gets a chance to double the bet for the next roll. Would you rather be player or dealer?

Imagine the result of any roll recorded as a three-digit number using the digits 1 to 6. Thus, 353 means that the first die showed 3, the second one 5, and the third one another 3. There are 6 possibilities for the first digit; for each of these, there are 6 possible second digits — giving a total of 36 patterns for the first two digits; for each of these, there are 6 possible third digits. All in all, that makes $6 \times 36 = 216$ *possibilities.*

Suppose, the player bet on 6. Then the dealer wins if the record of the roll contains only the digits 1 to 5. By the above counting scheme, that happens in $5 \times 25 = 125$ *of the 216 possible cases. So, the odds are 125 to 91 for the dealer to win.*

Admittedly, the player gets a bonus of the stake (or even twice that) every time the number 6 comes up more than once. However, I think that this does not happen often enough to off-set the player's basic disadvantage.

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Reflections on Hari Koirala's Thesis.

Hari Koirala's very interesting study looks at 16 prospective secondary school teachers, with fairly strong university backgrounds in mathematics (including some probability theory), and examines their responses to 3×5 elementary questions involving chance. It concludes that, in this respect, their university training has failed them (half the subjects agree with this conclusion), and recommends that the university put a greater emphasis on the conceptual development of its students in basic mathematical probability. I fully agree with this recommendation.

So far so good. My difficulties lie in trying to understand the way the study moves from its data to its conclusion. It does so by classifying the responses as I,F,I/F,and I↔F, where F means "formal" and I, "informal". A strict definition of "formal" is to be found on p.33:

"If someone uses probability axioms, definitions, formulas, rules, and theorems directly from their school or university courses they will be classified as formal conceptions... the use of a term such as chance, independence, or randomness is not enough... "

This definition is repeated, as a paraphrase, on p.56:

"Formal conceptions... refer to... use of mathematics concepts, principles, formulas, rules, and applications... used ... correctly or incorrectly... "

The results of the response classification are tabulated on p.58. The rows of the 16×15 matrix shown there are labelled by the names of the 16 respondents, the columns by the 15 tasks; the entries are I,F,I/F, or I↔F, the latter meaning that there was conflict between informal and formal conceptions in the respondent's mind.

Lumping together certain tasks which (deliberately, to check consistency) had similar mathematical contents, I boiled them down to 9 different questions, which are shown on the two-page document called "Informal Responses to Elementary Questions About Chance". As the title indicates, they are accomanpied by responses which are quite convincing but exhibit none of the characteristics of "formal" conceptions as defined above. In fact, in only three of the 9 questions do I see any substantial necessity for mathematical background to reveal itself.

These three questions correspond to the columns 4, 7, 10, and 14 of the table on p.58. In all the other tasks, a "formal" response would be somewhat construed — at any rate, it could (and pedagogically should) be avoided easily and naturally. Lest it be thought that my criteria are uncommon and specifically professional, I have written out the imaginary "informal" responses for all to see.

So, here is my difficulty: in only 4 of the 15 tasks, do I see any hope of being able to distinguish clearly between educated (i.e., "formal") and untutored but adequate responses. For instance, what Paul says on p.90 could easily have come from my maternal grandmother, who was an unschooled peasant. Nonetheless, Paul's response is marked F (as are all of his answers, except in the thumb-tack task, where everybody scores I/F). Is it because he twice says "independent"? Remember: the simple use of a term is not enough...

I do not disagree with Hari's conclusion. Indeed, the evidence contained in the dissertation makes me suspect that I would arrive at the same. However, I would probably get there by a different route: by tracking down the inadequacies of the responses marked "informal": vagueness, evasiveness, superstition, and downright fallacies (like the Gambler's Fallacy or the Representativeness Heuristic).