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Fibonacci Type Decimal Expansions.

1. Rappels. It was noted long ago, and Tony Gardiner reminded Mathed of it last year, that the decimal expansion of $1/89 = 0.01123595505617977528089887640449 4\ldots$ followed the Fibonnacci Sequence, if it is read as $0.0112358 + 0.00000013 + 0.000000021 + 0.0000000034 + \ldots$

To see how that comes about, consider formal power series $f(X) = \sum_{k=0}^{\infty} a_k X^k$ such that $a_k = a_{k-1} + a_{k-2}$ for all $k > 2$. This recursion relation easily implies the equation

$$f(X)(1 - X - X^2) = a_1 X + (a_2 - a_1)X^2$$

whence $f(1/z) = (a_1 z + a_2 - a_1)/z^2 - z - 1$. The reason for specialising X to z^{-1} is to prepare for numerical expansions in terms of 10^{-1}.

Thus, with $z = 10$, putting $a_1 = 0$ and $a_2 = 1$ yields Tony's formula for 1/89. But there is no need to start at the beginning: taking $a_1 = 5$ and $a_2 = 8$ reveals that $53/89 = 0.595505617977528089887640449 4382\ldots$ This is just a final segment of the expansion for 1/89, but since the latter is periodic, it is also a *shift* — by 5 places (count them!), i.e. multiplication by 10^5, whence $10^5 = 53$ modulo 89. ...

Of course, there is also no need to stick with the classical Fibonacci sequence. Why not try the Lucas sequence ($a_1 = 1, a_2 = 3$)? It yields 12/89 — nothing essentially new.

2. Variations. To get something really different, we follow our friend Fred Harwood, and "parse" the decimals two, three, or more places at a time, by setting $z = 100, 1000, \ldots$. Fred discovered this while computing $1/19 = 0.052631578947368421\ldots$ whose period has length 18. Its structure can be more clearly recognised when it is written as $0.05263157 + 88 \times 10^{-10} + 145 \times 10^{-12} + 233 \times 10^{-14} + 378 \times 10^{-16} + 611 \times 10^{-18} + 1600 \times 10^{-20} \ldots$

Defining $d(n) = 100^n - 10^n - 1$, we get $d(2) = 9899$, $d(3) = 998999$, etc. The prime factorisations of these numbers are:

$$\begin{aligned} d(2) &= 19 \times 521 \\ d(3) &= 179 \times 5581 \\ d(4) &= 29 \times 3447931 \\ d(5) &= 178439 \times 56041 \end{aligned}$$

and so on — something to experiment with. Are there always just 2 factors? Wherever the relation $a_k = a_{k-1} + a_{k-2}$ comes up, so necessarily does $z^2 = z + 1$ and therefore the Golden Ratios and $\sqrt{5}$, their sum. By quadratic reciprocity, a prime for which 5 is a quadratic residue must be congruent ± 1 modulo 5, i.e. have 1 or 9 as the last digit. But couldn't we have 3 nines, for instance?