The Duisburg Interviews

In the autumn of 2002, I had the privilege of interviewing small groups of secondary students in and around Duisburg. In order to summarize what I learnt in these interviews, I am imagining a fourth year student, Tom, who is taking a didactics course because he wishes to become a teacher. All this is happening at the well-known University of Schwetzingen in Germany. His assignment is to write a short essay with the title "Was sind and was sollen die Zahlen?". He is allowed to write it in English, in order to keep up his mother tongue – his father was in the US Army, stationed in Germany.

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WHAT ARE NUMBERS, AND WHAT ARE THEY FOR?

by Tom Sawyer

Numbers are words, either spoken or written, with which we count and measure objects in our environment, as well as relationships between such objects. We use the decimal system, which I need not explain here, except to make two points:

(i) numbers are *finite* strings of digits, punctuated by a single dot;
(ii) numbers allow measurements of arbitrary size and precision.

Since this essay is about several kinds of numbers, I shall call those just defined the *reasonable* numbers. They are the ones described in 1585 by Simon Stevin in his booklet "De Thiende". By shifting the decimal point (and keeping track of the shift) they are easily related to integers, and can be added, subtracted, and multiplied just like the latter. They are the only numbers you ever see displayed by a calulator (or computer), and they are the only numbers you ever need in even the most precise applications.

However, things are not as simple when you come to division. There is no difficulty in dividing by powers of 2 or 5, and I remember that this is all we ever did in elementary school. Either that or we were left with a remainder which could be made as small as desired. For example, 1 divided by 3 is 0.333333333 with a of remainder of 0.000000001. If our 1 were the circumference of the earth, that remainder would be 4 mm. Is that close enough?

For some people it is not: they want zero-tolerance, no remainder at all but an *exact* quotient for any two reasonable numbers (or integers, if you prefer), and are thus led make up a kind of unreasonable number which they call "rational". In decimal language, such a thing would be an *infinite* strings of digits, with the only saving grace of being *periodic*, i.e., from some point onward repeating the same string, like 547.23987598759875 ... literally *ad infinitum.*

Other people (or maybe the same) are bothered by the fact that a number which is reasonable in the decimal system may be unreasonable in another one, e.g., 0.2 (dec) = 0.333333...(hex). If you always work with these infinite periodic beasts, you can be sure that you are actually handling quotients of integers, which are periodic in *any* system. That is very nice esthetically, but is it worth the (usually futile) effort of grasping "infinity" in Grade 9? At that age, most people are naive enough to think that words must have meanings – and thus get confused.

When I was in Grade 12, our teacher invited some Canadian professor to interview some of us about rational and irrational numbers – the latter being represented by infinite, non-periodic strings (try to imagine that, honestly!). Here are some of the misgivings I recall arising in our group.

1) The strange double standard, or so it seemed, we were taught to describe asymptotic behavior. In calculus it was correct to say: as x goes to infinity, the function $f(x) = 1-(1/10)^x$ approaches the constant function $g(x) = 1$ but never meets it. However, on the number line, 0.99999... always does meet 1, in fact, it *equals* 1, even though the terms defining this sequence are exactly the values $f(n)$ for positive integer n.

2) The periodicity business. The number 144/233=0.618025751073... has a period of length 232 (imagine it written out!), while the famous Golden Mean = 0.61803398875... has no period at all. Stare at the first 200 places on the right hand side of both and decide which one is rational! Since long periods outnumber short ones by a long way, such questions are usually unanswerable, and therefore of no practical importance.

3) The role of fractions, roots, etc. as "place holders". The professor wrote two expressions made up of a 2 and a 7 on the board. In the first one, the digits were separated by a horizontal bar, in the second one, by the familiar zig-zag denoting "root". He asked for qualitative differences. We discussed this for a while and then agreed that both were place holders – separated from actual numbers by some computational or conceptual steps. You could work with those place-holders (we had learned the calculus of fractions, and had played a bit with identities involving roots) but they were not the real thing – for instance, you cannot easily size them up, i.e., compare them.

The interview was quite extensive and very interesting. When we were asked about the relative frequency of rationals versus irrationals or short periods (among rationals) versus long ones, we quickly came to the right conclusion. I was surprised later, that some other class mates had that all backwards. One even said that irrationals were only "stop-gaps" and very rare. But we were not exactly brilliant either, e.g., I remember saying that π was irrational because the circle had infinitely many corners...

In the meantime, I have learned some number theory and some real analysis. I like both (the former more than the latter) and no longer have anything against rationals or irrationals. But in school, I would have gained more insight if we had stuck to the good old Stevin numbers – and I would have lost nothing. After all, they were using log and trig tables (with six or eight places) for centuries, and Herr Dedekind could have defined his cuts on Stevin's number line just as easily as on the chaotic rationals – and obtained the same "real" numbers (which are basically line-segments in disguise).

Looking back at items 1 – 3 above, I now realize that the "double standard" is due to the (inevitably) casual language of school mathematics. I still think that the periodicity business is given too much importance, and that the calculus of fractions has nothing to do with integers: it is a formalism which also allows you to derive the Addition Law of "tan" from that of "sin" (for instance). Our reluctance to recognize $\sqrt{7}$ as an actual number reminds me of the trick question "why must you always integrate *counter-clockwise* in order for the Residue Theorem to be true?"