Introduction to Matrices — Outline.

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Elementary matrix theory has three facets (which, of course, overlap in multiple ways):

(A) *Elimination, inversion, linear equations, dependence, bases. Determinants.*
(B) *Eigenvectors, minimal and characteristic polynomials, similarity, standard forms.*
(C) *Inner product, orthogonality, isometries, singular values, spectral theorem.*

Note: as far as scalars are concerned, (A) is rational, (B) requires algebraic extensions, and (C) needs real (or complex) numbers.

These facets will be explored in that order (I) in Chapters 1, 2, 5, (II) again in Chapters 6, 7, 9, and finally (III) in Chapters 10, 11, 12. Parts I and II concentrate, respectively, on two and three dimensions — which have enough special features to derserve separate treatment. Chapters 3, 4, and 9, marked by asterisks, are optional and deal with applications.

Of the three facets, (B) will be given the largest spread in Part I, because it is already interesting but still easily manageable in two dimensions. The most geometric item (C) reaches the right balance of challenge versus tractability in three dimensions, and will therefore be stressed in Part II. The main focus in Part III will be on (A), which is conceptually the simplest of the three.

1. Basic matrix operations. Product, inverse, determinant of 2×2-matrices; singularity vs. invertibility. Geometric view of matrix action, typical examples. Supplement: arcs of angles

2. Eigenvectors and similarity. Characteristic equation, diagonalization; three canonical forms for 2×2 matrices. Supplement: complex numbers.

3*. Discrete dynamical systems. Powers of transition matrices; possible patterns of orbits. Linear difference equations with applications. Supplement: non-linear systems.

4*. Linear differential equations. Matrix exponentials, solution curves, applications; power series. Supplement: angles and logarithms (differential vs. functional equations).

5. Dot product and matrix geometry. Dot and transpose; symmetric and orthogonal matrices in the plane; singular values. Dot in higher dimensions. Lines and hyperplanes.

6. Elimination and elementary matrices. Prologue in the plane. Invertibility in three dimensions; LU-factorization. Determinants (3×3) and cross products. Volumes and areas.

7. Diagonalizability. Characteristic polynomial, multiplicity of eigenvalues; eigenlines and eigenplanes in 3 dimensions. Supplement: standard forms, Cayley-Hamilton, and the minimal polynomial.

8. Symmetric and orthogonal matrices. Spectral theorem ($\dim \leq 3$), rotations and reflections in 3-space. Supplement: rotations in computer graphics.

9*. Quadratic polynomials in three variables. Conics and quadrics; completing the square; principal axes of quadratic forms. Supplement: the second derivative of a multivariate function.

10. Independence and dimension. Fundamental theorem of square matrices; dimension. Row space and column space; bases via elimination. Rank vs. nullity. Supplement: higher determinants.

11. Similarity and standard forms. Minimal polynomials, eigenvalues. Diagonalization and triangulation. Supplement: the Jordan form of a triangular matrix.

12. Orthogonality and orthogonalization. Spectral theorem and polar decomposition. The Gram-Schmidt process. Supplement: least squares, linear regression, and curve fitting.